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Semi-Separated Conditions for Almost Periodic Solutions

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INTRODUCTION

The notion of separated solutions of a differential equation was introduced by Amerio [1] in order to give sufficient conditions for the existence of an almost periodic solution to an almost periodic equation. We introduce the concept of semiseparated and prove two Amerio type theorems. Both have several novel features. In particular, one may be stated without the introduction of the equations in the hull. The ideas in this paper were motivated by the papers of Seifert [2] and Yoshizawa [3]. Yoshizawa has shown how many stability criterion yield asymptotically almost periodic solutions. Seifert introduces a new necessary and sufficient condition for almost periodicity. We show how an analogous condition is equivalent to asymptotic almost periodicity. This condition leads naturally to the notion of semiseparated and our other results.

All functions will be vector valued and $|\cdot|$ will denote any norm.

ASYMPTOTIC ALMOST PERIODICITY

The notion of asymptotic almost periodicity was introduced by Frechet [4]. Although we have need for only two, we will list five alternative descriptions of such functions. A continuous function f on R is asymptotically almost periodic ($=$ a.a.p.) on R^+ if and only if one of the following five conditions is satisfied:

(i) $f(t) = p(t) + q(t)$, where p is almost periodic and $\lim_{t\to\infty} q(t) = 0$;

(ii) given $\epsilon > 0$ there exists T and a set E, relatively dense in R, such that $\sup\{|f(t+\tau) - f(t)|;\ t \geqslant T,\ t+\tau \geqslant T\} < \epsilon$ for each $\tau \in E$;

(iii) the same as (ii) except E is only required to be relatively dense in $R^+ = [0, \infty)$;

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(iv) for any sequence $\alpha_n' \to \infty$, there exists a subsequence $\{\alpha_n\}$ such that $\lim_{n\to\infty} f(t + \alpha_n)$ exists uniformly on any interval of the form $[\beta, \infty)$, $\beta > -\infty$;

(v) the same as (iv) except that $\beta = 0$ is the only left end point considered.

We shall add one condition to this list. We introduce the operator notation for taking limits of translates of functions. If $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence, then $T_\alpha f \equiv \lim_{n\to\infty} f(t + \alpha_n)$ provided the limit exists. The mode of existence will be specified in each instance. If $\alpha' = \{\alpha_n'\}_{n=0}^{\infty}$ is a subsequence of α we write $\alpha' \subset \alpha$. Also $\alpha + \beta$ will denote $\{\alpha_n + \beta_n\}_{n=0}^{\infty}$. By $\alpha > 0$ we mean $\alpha_n > 0$ for each n. If $\alpha \subset \alpha'$ and $\beta \subset \beta'$, then α and β are said to have matching subscripts if $\alpha = \{\alpha'_{n_k}\}$ and $\beta = \{\beta'_{n_k}\}$.

Definition. A function f is said to satisfy the Seifert semisplitting condition ($= S^3C$) on R^+ if given a sequence γ' with $\lim \gamma_n' = \infty$, there exists $\gamma \subset \gamma'$ and a number $d(\gamma) > 0$ such that $T_\gamma f$ exists pointwise and if α is a sequence with $\alpha > 0$, $\beta' \subset \gamma$, $\beta'' \subset \gamma$ are such that $T_{\alpha+\beta'} f = g$ and $T_{\alpha+\beta''} f = h$ exist pointwise, then either $g \equiv h$ or $|g(t) - h(t)| \geqslant 2d(\gamma)$ for all $t \in R^+$.

This definition is inspired by Condition A of Seifert [2] and the main results it leads to, Theorems 1 and 2 below. The "operator" $T_{\alpha+\beta}$ as β runs through subsequences of γ "splits" f into functions which are a positive distance apart on R^+, hence the name. Seifert's Condition A requires this on R.

Theorem 1. *Let f be continuous. Then f is* a.a.p. *if and only if f satisfies S^3C.*

Proof. Let f satisfy S^3C. We will show that condition (v) holds. Let γ' be a sequence such that $\lim \gamma_n' = \infty$. There exists, by S^3C, $\gamma \subset \gamma'$ such that $T_\gamma f$ exists pointwise. If the convergence is not uniform on R^+, then there exist sequences $\delta' > 0$, $\alpha' \subset \gamma$, $\beta' \subset \gamma$ and $\epsilon > 0$ such that $|f(\alpha_n' + \delta_n') - f(\beta_n' + \delta_n')| \geqslant \epsilon$ where we may pick $\epsilon < d(\gamma)$. Since $T_\gamma f(0)$ exists we have $|f(\alpha_n') - f(\beta_n')| < d(\gamma)$ for large n. Consequently, $k(t) = f(t + \alpha_n') - f(t + \beta_n')$ satisfies $|k(0)| < d(\gamma)$ and $|k(\delta_n')| \geqslant \epsilon$ for large n. Thus there exists δ_n'' such that $\epsilon \leqslant |k(\delta_n'')| < d(\gamma)$. Consider the sequences $\alpha' + \delta''$ and $\beta' + \delta''$. By S^3C there exist sequences $\alpha + \delta \subset \alpha' + \delta''$ and $\beta + \delta \subset \beta' + \delta''$ with matching subscripts such that $T_{\alpha+\delta} f = g$ and $T_{\beta+\delta} f = h$ exist pointwise, and $g \equiv h$ or $|g(t) - h(t)| \geqslant 2d(\gamma)$ on R^+. But $|g(0) - h(0)| = \lim |f(\alpha_n + \delta_n) - f(\beta_n + \delta_n)|$ so that $0 < \epsilon \leqslant |g(0) - h(0)| \leqslant d(\gamma)$. This contradiction shows that $T_\gamma f$ exists uniformly on R^+. Conversely, if f is a.a.p., let γ' be given with $\lim \gamma_n' = \infty$. Using condition (i), let $f = p + q$. There exists $\gamma < \gamma'$ such that $T_\gamma p = k$ exists uniformly on R. Take $d(\gamma) = 1$. Let $\alpha > 0$ and $\beta' \subset \gamma$, $\beta'' \subset \gamma$ such that $T_{\alpha+\beta'} f = g$ and $T_{\alpha+\beta''} f = h$ exist pointwise. We may take subsequences so that these exist uniformly on R^+ and so that $T_\alpha k$ exists uniformly on R (k is a.p.). Then $g = T_{\alpha+\beta'} p$ and $h = T_{\alpha+\beta''} p$. By a theorem of Bochner [5], see Condition B of [2], we can extract subsequences $\alpha' \subset \alpha$, $\bar{\beta}' \subset \beta'$, $\bar{\beta}'' \subset \beta''$ with matching subscripts so that $g = T_{\alpha'+\bar{\beta}'} p = T_{\alpha'} T_{\bar{\beta}'} p = T_{\alpha'} T_\gamma p = T_{\alpha'} k = T_\alpha k$ and $h = T_{\alpha'+\bar{\beta}''} p = T_{\alpha'} T_{\bar{\beta}''} p = T_{\alpha'} T_\gamma p = T_{\alpha'} k = T_\alpha k$ so that $g = h$. Hence f satisfies S^3C.

We notice the rather curious fact that is shown by the above proof. If f satisfies S^3C then the last condition never holds, i.e., $g = h$ always and there is never any "splitting".

Semiseparated Solutions

The Condition A of Seifert [2] which inspired our S^3C is precisely what is used to prove Amerio's theorem on separated solution. It is thus natural to introduce a notion that is similarly related to S^3C. We succeed in proving Amerio's theorem with this weaker hypothesis. In this section, K denotes a fixed compact set in R^n.

Definition. A function φ is in K if $\varphi(t) \in K$ for all $t \in R$.

Definition. A solution φ of an equation is semiseparated in K, if there exists a number $d(\varphi) > 0$ such that if ψ is any other solution in K of the same equation, then $|\varphi(t) - \psi(t)| \geqslant d(\varphi)$ for $t \in R^+$.

Note that Amerio's separated condition just replaces R^+ by R in this definition.

We shall be considering an a.p. system

$$x' = F(t, x) \tag{1}$$

with $F(t, x)$ a.p. in t uniformly for $x \in K$. We let $H(F)$ be the set of functions of the form $T_\alpha F(t, x) = \lim F(t + \alpha_n, x)$ where the limit exists uniformly on $R \times K$. In the subsequent discussions we repeatedly use the following facts. If f is a.p. and α', β' are given there exist $\alpha \subset \alpha'$ and $\beta \subset \beta'$ with matching subscripts such that $T_{\alpha+\beta} f = T_\alpha(T_\beta f)$. This is the Bochner result alluded to above. Next, if $G \in H(F)$ then there exists α such that $\lim \alpha_n = \infty$ and $T_\alpha F = G$. If φ is a solution to $x' = F(t, x)$ in K and α' is given, then there exists $\alpha \subset \alpha'$ such that $T_\alpha \varphi$ exists uniformly on compact sets and is a solution to $x' = T_\alpha F(t, x)$ where $T_\alpha F$ exists uniformly. This is an Ascoli argument followed by a diagonalization argument. Finally, we use the fact that $G \in H(F)$ implies that $H(G) = H(F)$.

LEMMA 1. *Let $F(t, x)$ be* a.p. *uniformly for $x \in K$. Suppose that for each $G \in H(F)$, every solution of $x' = G(t, x)$ in K is semiseparated. Then for each equation in $H(F)$, the number of solutions in K is finite. Consequently, every equation in $H(F)$ has the same number of solutions in K and the separation constant d may be picked independent of the solutions.*

Proof. That each equation has only a finite number of solutions in K is a consequence of the compactness of K and the resulting compactness of the solutions in K. But no solution can be the limit of others by the semiseparated condition. Consequently, the number of solutions of any equation is finite and d may be picked as a function of the equation. Let $G, L \in H(F)$, and $T_{\alpha'}G = L$ with $\lim \alpha_n' = \infty$. If φ and ψ are two solutions of $x' = G(t, x)$ in K, let $\alpha \subset \alpha'$ such that $T_\alpha\varphi$ and $T_\alpha\psi$ exist uniformly on compacta and are solutions of $x' = L(t, x)$. Then $| T_\alpha\varphi - T_\alpha\psi | \geqslant d(G)$. Consequently, if $\varphi_1, \ldots, \varphi_n$ are the solutions of $x' = G(t, x)$ in K, then $T_\alpha\varphi_1, \ldots, T_\alpha\varphi_n$ are distinct solutions of $x' = L(t, x)$ in K such that $| T_\alpha\varphi_i - T_\alpha\varphi_j | \geqslant d(G)$ for $i \neq j$. Hence the number of solutions of $x' = L(t, x)$ in K is greater than or equal to n. By a symmetry argument the reverse is true, hence each equation has the same number. Thus the $T_\alpha\varphi_i$ exhaust the solutions of $x' = L(t, x)$ in K so that $d(G) \leqslant d(L)$. Again by symmetry, $d(G) = d(L)$.

THEOREM 2. *Let $F(t, x)$ be* a.p. *uniformly for $x \in K$ and suppose that each equation in $H(F)$ has only semiseparated solutions in K. If some equation in $H(F)$ has a solution in K, then every equation in $H(F)$ has a solution in K. All such solutions are* a.a.p. *and every equation in $H(F)$ has an* a.p. *solution in K.*

Proof. The first statement has been noted above. Let φ be a solution in K of $x' = G(t, x)$ for $G \in H(F)$. Let δ be the separation constant. The claim is that φ satisfies S^3C with $d = \delta/2$ for any γ' sequence. For let γ' be any sequence with $\lim \gamma_n' = \infty$ and $\gamma \subset \gamma'$ such that $T_\gamma G = L$ and $T_\gamma\varphi$ exist. Let $T_{\alpha+\beta'}\varphi = g$ and $T_{\alpha+\beta''}\varphi = h$ where $\beta' \subset \gamma$, $\beta'' \subset \gamma$, and $\alpha > 0$. Again take further subsequences with matching subscripts so that (without changing notation) $T_{\alpha+\beta'}G = T_\alpha T_{\beta'}G = T_\alpha T_\gamma G = T_\alpha L$ and similarly $T_{\alpha+\beta''}G = T_\alpha L$ so that g and h are solutions of the same equation and $g \equiv h$ or $| g(t) - h(t)| \geqslant \delta = 2d$ on R^+. Consequently φ satisfies S^3C and is a.a.p. To show that each equation has an a.p. solution, let φ be a solution of $x' = G(t, x)$ in K which by the above is a.a.p. Let $\alpha_n' = n$. There is $\alpha \subset \alpha'$ so that $T_\alpha G = L$ and $T_{-\alpha}L = G$ exist uniformly and $T_\alpha\varphi = \psi$ and $T_{-\alpha}\psi$ exist uniformly on compacta, and $T_{-\alpha}\psi$ is a solution to $x' = G(t, x)$. Using (i) it is easy to see that $\psi = T_\alpha p$ and thus $T_{-\alpha}\psi$ exists uniformly and is a.p.

INHERITED AND SEPARATING PROPERTIES

One of the weaknesses of Amerio type theorems is that one must consider the same hypothesis for every equation in the hull $H(F)$. A second weakness is the necessity for each solution in K to be separated from all other solutions in K. We propose partial remedies in our next two theorems. We will assume throughout that $F(t, x)$ is a.p. uniformly for $x \in K$ and that all solutions referred to are in K, K compact.

DEFINITION. A property P is inherited if when φ a solution of $x' = F(t, x)$ has property P with respect to the solutions of $x' = F(t, x)$ and $G = T_\alpha F$, $\psi = T_\alpha \varphi$ uniformly on compacta, then ψ has property P with respect to the solutions of $x' = G(t, x)$.

LEMMA 2. *Uniform stability is an inherited property if for every $G \in H(F)$, solutions of initial value problems of $x' = G(t, x)$ are unique.*

This is the content of Lemma 4 in Yoshizawa [3]. Here φ is uniformly stable if for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that for any solution ψ and any $t_0 \in R$, $|\varphi(t_0) - \psi(t_0)| < \delta(\epsilon)$ implies that $|\varphi(t) - \psi(t)| < \epsilon$ for all $t \geqslant t_0$.

DEFINITION. A property P of a solution φ in K of $x' = F(t, x)$ is a semiseparating property if for any other solution ψ with property P of $x' = F(t, x)$ in K there exists a $\delta(\varphi, \psi) > 0$ such that $|\psi(t) - \varphi(t)| \geqslant \delta(\varphi, \psi)$ for $t \in R^- = (-\infty, 0]$.

Several remarks are in order. Note that we intend the use of semiseparated here to be essentially the same as before but we here use R^- instead of R^+. We used R^+ before in order to conform to the usual definitions of a.a.p. Here we use R^- because it will turn out that the usual stability properties on R^+ imply semiseparatedness on R^-. It should be clear that all previous results with R^+ replaced by R^- also hold. (Consider the replacement of t by $-t$.) In particular a.a.p. on R^- means that in (i), $q \to 0$ as $t \to -\infty$. Finally notice that δ depends on both φ and ψ. We require no uniformity.

LEMMA 3. *Uniform stability is a semiseparating property in the presence of uniqueness of solutions of initial value problems.*

Proof. Let φ be a uniformly stable solution, and let ψ be any other solution. Set $2\epsilon = |\varphi(0) - \psi(0)|$. This is positive by uniqueness. If there is a $t_0 \in R^-$ such that $|\varphi(t_0) - \psi(t_0)| < \delta(\epsilon)$ then $|\varphi(0) - \psi(0)| < \epsilon$; consequently, $|\varphi(t) - \psi(t)| \geqslant \delta(\epsilon)$ on R^-.

We now establish a counterpart of Lemma 1.

LEMMA 4. *Suppose property P is inherited and is semiseparating. Suppose that $x' = F(t, x)$ has only a finite number of solutions in K with property P. Then every equation in $H(F)$ has the same number of solutions in K with property P and the separation constant $d(\varphi, \psi)$ may be picked to be independent of solution and equation.*

Proof. Since we only consider a finite number of solutions of $x' = F(t, x)$ the separation constant for solutions of $x' = F(t, x)$ with property P may be picked independent of these solutions, say δ_F. Let φ, ψ be solutions of $x' = F(t, x)$ in K with property P and $G \in H(F)$. Let $T_\alpha F = G$, $T_\alpha \varphi$, and $T_\alpha \psi$ exist uniformly on compacta. Then $| T_\alpha \varphi - T_\alpha \psi | \geqslant \delta_F$. Furthermore, $T_\alpha \varphi$ and $T_\alpha \psi$ have property P with respect to solutions of $x' = G(t, x)$. So if $\varphi_1, \ldots, \varphi_n$ are the solutions of $x' = F(t, x)$ in K with property P, then $T_\alpha \varphi_1, \ldots, T_\alpha \varphi_n$ are distinct solutions of $x' = G(t, x)$ with property P. So $x' = G(t, x)$ has at least n solutions with property P. On the other hand, if $\psi_1, \ldots, \psi_m$ are solutions of $x' = G(t, x)$ with property P, by a similar construction $T_\beta \psi_1, \ldots, T_\beta \psi_m$, $(T_\beta G = F)$, are solutions of $x' = F(t, x)$ with property P. Hence $m \leqslant n$. But $m \geqslant n$ by the previous argument. Thus $m = n$ and δ_G may be taken to be δ_F, since the $T_\alpha \varphi_i$ exhaust the solutions of $x' = G(t, x)$ with property P.

THEOREM 3. *Let $F(t, x)$ be* a.p. *uniformly for $x \in K$. Let P be an inherited property that is semiseparating. If $x' = F(t, x)$ has only a finite number of solutions in K with property P, then every such solution is* a.a.p. *in R^- and there exists an* a.p. *solution in K.*

Proof. Let φ be a solution in K with property P and let δ be the separation constant. We claim that φ satisfies S^3C with R^+ replaced by R^- and $d = \delta/2$. For let $\lim \gamma_n' = -\infty$. Pick $\gamma \subset \gamma'$ such that $T_\gamma F = G$ and $T_\gamma \varphi$ exist uniformly on compacta. Let $\alpha < 0$, $\beta' \subset \gamma$, and $\beta'' \subset \gamma$, such that $T_{\alpha+\beta'} \varphi = g$ and $T_{\alpha+\beta''} \varphi = h$ exist. Taking subsequences with matching subscripts and not changing notation we may assume that g and h are solutions to $T_\alpha T_\gamma F = T_\alpha G$ with property P. Hence $g \equiv h$ or $|g(t) - h(t)| \geqslant \delta = 2d$ on R^-. Hence φ is a.a.p. on R^-. As before, we get an a.p. solution.

We have an immediate application of Lemmas 2 and 3 and Theorem 3.

THEOREM 4. *Let $F(t, x)$ be* a.p. *uniformly for $x \in K$. Suppose that for each $G \in H(F)$, solutions of initial value problems for $x' = G(t, x)$ are unique. If $x' = F(t, x)$ has a finite number of solutions in K which are uniformly stable, then each such solution is* a.a.p. *on R^- and there is an* a.p. *solution in K.*

The special case when there is only one solution that is uniformly stable is a stronger version of Theorem 3 of [6] where it is required that this uniformly stable solution be the only solution in K.

The fact that uniform asymptotic stability is a separating condition has been noted in several places, for example in [7, p. 143]. The notion of inherited property, though not formally introduced, is used in Yoshizawa [3]. In fact, many of his results are of the form Property P is inherited, or Property P is separating. He uses stability only on R^+ so that our results do not overtly overlap his. His paper also gives references to other theorems which may be proved in our way.

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